THE RING OF REGULAR FUNCTIONS OF AN ALGEBRAIC MONOID

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Abstract. Let M be an irreducible normal algebraic monoid with unit group G. It is known that G admits a Rosenlicht decomposition, $G = G_{\mathrm{ant}}G_{\mathrm{aff}} \cong (G_{\mathrm{ant}} \times G_{\mathrm{aff}})/G_{\mathrm{aff}} \cap G_{\mathrm{ant}}$, where G_{ant} is the maximal anti-affine subgroup of G, and G_{aff} the maximal normal connected affine subgroup of G. In this paper we show that this decomposition extends to a decomposition $M = G_{\mathrm{ant}}M_{\mathrm{aff}} \cong G_{\mathrm{ant}}*_{G_{\mathrm{aff}}\cap G_{\mathrm{ant}}} M_{\mathrm{aff}}$, where M_{aff} is the affine submonoid $M_{\mathrm{aff}} = \overline{G_{\mathrm{aff}}}$. We then use this decomposition to calculate $\mathcal{O}(M)$ in terms of $\mathcal{O}(M_{\mathrm{aff}})$ and $G_{\mathrm{aff}}, G_{\mathrm{ant}} \subset G$. In particular, we determine when M is an anti-affine monoid, that is $\mathcal{O}(M) = \Bbbk$.

1. Introduction

The theory of *affine* algebraic monoids has been investigated extensively over the last thirty years. See [11, 12, 18] for different accounts of these developments. More recently there has been some important progress on the structure of non-affine algebraic monoids. By generalizing a classical theorem of Chevalley, the authors of [6] prove that any normal algebraic monoid is an extension of an affine algebraic monoid by an abelian variety. This allows one to analyze the structure of such monoids in terms of more basic objects: affine monoids, abelian varieties and anti-affine algebraic groups.

To state our results we first introduce some notation. Let $\Bbbk$ be an algebraically closed field. We work with algebraic varieties X over $\Bbbk$, that is, integral, separated schemes over $\Bbbk$. An algebraic group is assumed to be a smooth group scheme of finite type over $\Bbbk$. If X is an algebraic variety we denote by $\mathcal{O}(X)$ the ring of regular functions on X. If X is an affine variety and $I \subset \mathcal{O}(X)$ is an ideal, we denote by $\mathcal{V}(I) = \big\{x \in X : f(x) = 0 \ \forall f \in I\big\}$; if $Y \subset X$ is a subset, we denote by $\mathcal{I}(Z) = \big\{f \in \mathcal{O}(X) : f(y) = 0 \ \forall y \in Y\big\}$. If X is irreducible we denote by $\Bbbk(X)$ the field of rational functions on X. If A is any integral domain we denote by $[A]$ its quotient field. Hence, if X is an irreducible affine variety then $\Bbbk(X) = \big[\mathcal{O}(X)\big]$.

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Let M be a connected, normal, *algebraic monoid* with unit group G (see Definition 2.1 below). The original motivation for this paper was to investigate the following basic question, first posed by M. Brion.

"How does one describe $\mathcal{O}(M)$, and when is it finitely generated?"

Although we do not answer this question completely, we obtain many remarkable results about $\mathcal{O}(M)$.

Let M and G be as above. By the results of [6], if $\alpha_G : G \to A$ is the unique Albanese morphism of G such that $\alpha_G(1_G) = 0_A$ (note the additive notation for A), then there exists a unique morphism $\alpha_M : M \to A$ such that $\alpha_M|_G = \alpha_G$. Furthermore, α is an affine morphism, and the scheme-theoretic fibers of α are normal varieties. The fiber at $1 \in A$ is M_{aff}, the unique irreducible, affine submonoid of M with unit group G_{aff}, the kernel of α. See Theorem 2.2 below.

The purpose of this paper is three-fold. First we identify $\mathcal{O}(M)$ in terms of the structure of M and M_{aff}, see Theorem 3.1. We then identify conditions under which $\big[\mathcal{O}(M)\big] = \big[\mathcal{O}(G)\big]$, see Theorem 3.15. Finally, we determine the conditions under which $\mathcal{O}(M) = \Bbbk$ (that is when M is an *anti-affine* algebraic monoid, Theorem 3.19). In order to establish our results we define the notion of a *stable* algebraic monoid (see Definition 3.9).

To obtain our main results we make use of the generalized Chevalley decomposition presented in [6], which states that, if M is an irreducible monoid then

$$M \cong G *_{G_{\mathrm{aff}}} M_{\mathrm{aff}},$$

where G_{aff} is the smallest affine algebraic group such that G/G_{aff} is an abelian variety (see Theorem 2.2 below). This structural result allows us to present a Rosenlicht decomposition $M = G_{\mathrm{ant}} *_{G_{\mathrm{aff}} \cap G_{\mathrm{ant}}} M_{\mathrm{aff}}$ that generalizes the corresponding decomposition $G = (G_{\mathrm{ant}} \times G_{\mathrm{aff}})/(G_{\mathrm{aff}} \cap G_{\mathrm{ant}})$ of G, where G_{ant} is the largest *anti-affine* subgroup of G. See [5] and Proposition 2.12 below. We then use this decomposition (of M) in Theorem 3.1 to calculate $\mathcal{O}(M)$ in terms of $\mathcal{O}(M_{\mathrm{aff}})$ and $H = G_{\mathrm{aff}} \cap G_{\mathrm{ant}}$.

Next we identify a set of central idempotents e of M such that $\mathcal{O}(M) = \mathcal{O}(eM)$. But such an idempotent can be chosen so that eM is a *stable* monoid. Consequently, we reduce ourselves to the study of stable monoids, thereby obtaining a characterization of the algebraic monoids M such that $\mathcal{O}(M) = \Bbbk$, the *anti-affine* algebraic monoids. See Theorem 3.19.

Let M be an anti-affine algebraic monoid and let $e \in E(M)$ be the minimum idempotent of M. In Theorem 3.21 we show that the retraction $\ell_e : M \to eM$, $\ell_e(m) = em$, is Serre's universal morphism from M to a commutative algebraic group (see [17, Thm. 8]).

We conclude the paper with Theorem 3.22. Here we show that there is an analogue of the Rosenlicht decomposition for a large class of normal, algebraic monoids. In particular the fibre $\varphi^{-1}(1)$, of the canonical map $\varphi : M \to Spec(\mathcal{O}(M))$, is an anti-affine monoid which we identify explicitly in terms of the internal structure of M.

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2. Preliminaries

In this section we assemble some of what is known about algebraic monoids with nonlinear unit groups. These results are due to M. Brion and the second named author [3, 6, 4, 15].

Definition 2.1. An *algebraic monoid* is an algebraic variety M together with a morphism $m : M \times M \to M$ such that m is an associative product and there exists a neutral element $1 \in M$. The *unit group* of M is the group of invertible elements

$$G(M) = \left\{ g \in M : \exists\, g^{-1}\,,\, gg^{-1} = g^{-1}g = 1 \right\}.$$

We denote the *set of idempotent elements* by $E(M) = \{e \in M : e^2 = e\}$.

It has been proved that $G(M)$ is an algebraic group, open in M (see for example [14]). The structure of M is significantly influenced by the structure of $G(M)$. For instance, the $G(M) \times G(M)$ action by left and right multiplication, $(a, b) \cdot m = amb^{-1}$, has $G(M)$ as open orbit and, if $G(M)$ is dense in M, an unique closed orbit, which is the *Kernel* of M, i.e. the minimum, closed, nonempty subset I such that $MIM \subset I$.

Recall that if G is algebraic group, then the Albanese morphism $p : G \to A(G)$ fits into an exact sequence

$$1 \longrightarrow G_{\mathrm{aff}} \longrightarrow G \xrightarrow{p} A(G) \longrightarrow 0$$

where G_{aff} is a normal connected affine algebraic group (since the group $\mathcal{A}(G)$ is commutative, its law will be denoted additively). Moreover, G_{aff} is the smallest affine algebraic subgroup such that G/G_{aff} is an abelian variety. This structure theorem is originally due to Chevalley, but now there is a modern proof in [7]. Recently it has been generalized from groups to monoids. The following theorem is a summary of this development.

Theorem 2.2 (Brion, Rittatore [3, 6, 4, 15])**.** *Let M be a normal irreducible algebraic monoid with unit group G. Then M admits a Chevalley decomposition:*

$$\begin{tikzcd}
1 \arrow[r] & M_{\mathrm{aff}} = \overline{G_{\mathrm{aff}}} \arrow[r] & M = \overline{G} \arrow[r, "p"] & A(G) \arrow[r] & 0 \\
1 \arrow[r] & G_{\mathrm{aff}} \arrow[r] \arrow[u, hook] & G \arrow[r, "p"'] \arrow[u, hook] & A(G) \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

where $p : M \to A(G) = G/G_{\mathrm{aff}}$ and $p|_G : G \to A(G)$ are the Albanese morphisms of M and G respectively, and M_{aff} is an affine algebraic monoid. $\square$

Definition 2.3. Let G be an algebraic group, and let a closed subgroup $H \subset G$ act on an algebraic variety X. The *induced space* $G *_H X$ is defined as the geometric quotient of $G \times X$ under the H-action $h \cdot (g, x) = (gh^{-1}, h \cdot x)$.

Under mild conditions on X (e.g. X normal and covered by quasi-projective H-stable open subsets), this quotient exists. Clearly, $G *_H X$ is a G-variety, for the action induced by $a \cdot (g, x) = (ag, x)$. We will denote the class of (g, x) in $G \times X$ by $[g, x] \in G *_H X$. The fundamental properties of $G *_H X$ were established by Bialynicki-Birula in [1]. See also [19].

Remark 2.4. Let G be an algebraic group, and $H \subset G$ a closed subgroup acting over an algebraic variety X. Then $\pi : G *_H X \to G/H$, which is induced by $(g, x) \mapsto gH$, is a fiber bundle over G/H, with fiber isomorphic to X.

Theorem 2.5 (Brion, Rittatore, [3, 6])**.** *Let M be a normal algebraic monoid and let Z^0 be the connected center of G. Then $A(G) \cong Z^0/(Z_0 \cap G_{\mathrm{aff}})$ and*

$$M = GM_{\mathrm{aff}} = Z^0 M_{\mathrm{aff}} \cong G *_{G_{\mathrm{aff}}} M_{\mathrm{aff}} \cong Z^0 *_{Z_0 \cap G_{\mathrm{aff}}} M_{\mathrm{aff}}.$$

Definition 2.6. If M is an algebraic monoid with unit group, we define the *center of M* be

$$\mathcal{Z}(M) = \big\{ z \in M : zm = mz \ \forall\, m \in M \big\},$$

the set of central elements.

It is clear that $\mathcal{Z}(M)$ is a closed submonoid of M, with unit group $G\big(\mathcal{Z}(M)\big) = \mathcal{Z}(G)$, the center of G. However, one should be aware that this monoid is not necessarily connected. Moreover, the following example shows that the $\mathcal{Z}(G)$ is not necessarily dense in $\mathcal{Z}(M)$.

Example 2.7. Let $r, s \in \mathbb{N}$, $r \neq s$, and consider the affine algebraic monoid

$$N = \big\{ \left(\begin{smallmatrix} t^r & a \\ 0 & t^s \end{smallmatrix}\right) : t, a \in \Bbbk \big\}.$$

Then $G(N) = \big\{ \left(\begin{smallmatrix} t^r & a \\ 0 & t^s \end{smallmatrix}\right) : t \in \Bbbk^*,\ a \in \Bbbk \big\}$. The center of $G(N)$ is the finite subgroup $\mathcal{Z}(G) = \big\{ \left(\begin{smallmatrix} t^r & 0 \\ 0 & t^s \end{smallmatrix}\right) : t^r = t^s \big\}$. The center of N is $\mathcal{Z}(N) = \mathcal{Z}(G) \cup \{0\}$. However $\overline{\mathcal{Z}(G)} \neq \mathcal{Z}(N)$.

In particular the zero matrix is a central idempotent which does not belong to $\overline{\mathcal{Z}(G)}$.

In what follows we collect some results about the *Rosenlicht decomposition* of an algebraic group G. This decomposition depicts G as the product of two subgroups, one affine and the other anti-affine. We refer the reader to [16], [8, Sec. III.3.8] and [5] for proofs and further results about this decomposition.

Definition 2.8. A connected algebraic group G is *anti-affine* if $\mathcal{O}(G) = \Bbbk$.

Remark 2.9. It is easy to see that any anti-affine group is commutative. See for example [5, Lem. 1.1].

Theorem 2.10 (Rosenlicht decomposition)**.** *Let G be a connected algebraic group. Then $\mathcal{O}(G)$ is a finitely generated algebra, in such a way that* $\operatorname{Spec}\big(\mathcal{O}(G)\big)$ *is an affine algebraic group.*

Let $\varphi_G : G \to \operatorname{Spec}\big(\mathcal{O}(G)\big)$ be the canonical morphism (the affinization*). Then $G_{\mathrm{ant}} = \operatorname{Ker}(\varphi_G)$ is a connected subgroup, contained in the center of G. The subgroup G_{ant} is the largest anti-affine subgroup scheme of G. Equivalently, G_{ant} is the smallest normal subgroup scheme of G such that G/G_{ant} is affine.*

Moreover, $G = G_{\mathrm{aff}}G_{\mathrm{ant}} \cong (G_{\mathrm{aff}} \times G_{\mathrm{ant}})/(G_{\mathrm{aff}} \cap G_{\mathrm{ant}})$, and $G_{\mathrm{aff}} \cap G_{\mathrm{ant}}$ contains $(G_{\mathrm{ant}})_{\mathrm{aff}}$ as an algebraic group of finite index. $\square$

In loose terms, an anti-affine algebraic group G is a non-split extension of an abelian variety by an affine, commutative algebraic group. See [5, Thm. 2.7]. But in the case $\operatorname{char}(\Bbbk) = p > 0$ the situation is considerably less complicated. Indeed we have the following simplifying result. See [5, Prop. 2.2].

Proposition 2.11 (Brion [5])**.** *Let G be an anti-affine, connected algebraic group over an algebraically closed field $\Bbbk$. If $\operatorname{char}(\Bbbk) = p > 0$ then G is a semi-abelian variety. i.e. G is the extension of an abelian variety by an affine torus group.*

$$1 \longrightarrow T \longrightarrow G \xrightarrow{p} A(G) \longrightarrow 0.$$

The following result generalizes Rosenlicht's decomposition to the case of algebraic monoids. It is essential for determining the ring of regular functions on an algebraic monoid.

Proposition 2.12 (Rosenlicht decomposition for M)**.** *Let M be a normal irreducible algebraic monoid with unit group G. Then $M \cong G_{\mathrm{ant}} *_{G_{\mathrm{aff}} \cap G_{\mathrm{ant}}} M_{\mathrm{aff}}$ so that*

$$M = G_{\mathrm{ant}}M_{\mathrm{aff}} \cong G_{\mathrm{ant}} *_{G_{\mathrm{aff}} \cap G_{\mathrm{ant}}} M_{\mathrm{aff}}.$$

Proof. Since $G = G_{\mathrm{ant}}G_{\mathrm{aff}}$, it follows that $M = GM_{\mathrm{aff}} = G_{\mathrm{ant}}M_{\mathrm{aff}}$. The morphism $G_{\mathrm{ant}} \times M_{\mathrm{aff}} \to M$ $(g, m) \mapsto gm$ induces a surjective morphism of algebraic monoids $\varphi : G_{\mathrm{ant}} *_{G_{\mathrm{aff}}\cap G_{\mathrm{ant}}} M_{\mathrm{aff}} \to M$, $\varphi\big([g, m]\big) = gm$. If $a, b \in G_{\mathrm{ant}}$ and $m, n \in M_{\mathrm{aff}}$ are such that $am = bn$, then $b^{-1}am = n$. It follows from Theorem 2.5 that $b^{-1}a \in G_{\mathrm{aff}} \cap G_{\mathrm{ant}}$, and thus $[a, m] = [b, n]$; that is, φ is injective. Since $\varphi|_G : G = G *_{G_{\mathrm{aff}}} G_{\mathrm{aff}} \to G$ is an isomorphism and that M is normal, it follows from Zariski's main Theorem that φ is an isomorphism. □

Corollary 2.13. *Let M be an irreducible, normal, algebraic monoid such that M_{aff} is a monoid with zero 0. Then $0M$ is the Kernel of M. Moreover, $0M$ is an algebraic group, and*

$$0M = 0G = 0G_{\mathrm{ant}} \cong G_{\mathrm{ant}}/(G_{\mathrm{aff}} \cap G_{\mathrm{ant}}).$$

Proof. By Proposition 2.12, we have that $0M = 0G_{\mathrm{ant}}M_{\mathrm{aff}} = 0G_{\mathrm{ant}}$. In particular, $0M$ is the image under the geometric quotient $G_{\mathrm{ant}} \times M_{\mathrm{aff}} \to M = G_{\mathrm{ant}} *_{G_{\mathrm{aff}}\cap G_{\mathrm{ant}}} M_{\mathrm{aff}}$ of the closed G_{ant}-orbit $G_{\mathrm{ant}} \times \{0\}$, and hence is closed in M. Since 0 is a central idempotent of M, it follows that $0M$ is the unique $(G \times G)$-closed orbit of M. Thus, by [14, Theorem 1], $0M$ is the Kernel of M.

Since 0 is central, it follows that $0M = 0G_{\mathrm{ant}}$ is an algebraic group. Consider the multiplication morphism $\ell : G_{\mathrm{ant}} \to 0G_{\mathrm{ant}}$, $\ell(g) = 0g$. Then

$$\ell^{-1}(0) = \{g \in G_{\mathrm{ant}} : 0g = 0\} = (G_{\mathrm{ant}})_0.$$

It follows that $g \in \ell^{-1}(0)$ if and only if $[g, 0] = [1, 0] \in G_{\mathrm{ant}} *_{G_{\mathrm{aff}}\cap G_{\mathrm{ant}}} M_{\mathrm{aff}}$; that is, if and only if $g \in G_{\mathrm{aff}} \cap G_{\mathrm{ant}}$. Since ℓ is a separable morphism, it follows that $0G_{\mathrm{ant}} \cong (G_{\mathrm{aff}} \cap G_{\mathrm{ant}})$. □

3. The algebra of regular functions of M

The following Theorem is the key to understanding the ring of regular functions on an algebraic monoid.

Theorem 3.1. *Let M be a normal algebraic monoid. Then $\mathcal{O}(M) \cong \mathcal{O}(M_{\mathrm{aff}})^{G_{\mathrm{aff}}\cap G_{\mathrm{ant}}}$.*

Proof. By Proposition 2.12, it follows that $M \cong G_{\mathrm{ant}} *_{G_{\mathrm{aff}}\cap G_{\mathrm{ant}}} M_{\mathrm{aff}}$, and hence $G_{\mathrm{ant}} \times M_{\mathrm{aff}} \to M$ is a geometric quotient. Thus

$$\begin{aligned} \mathcal{O}(M) = \mathcal{O}\big(G_{\mathrm{ant}} \times M_{\mathrm{aff}}\big)^{G_{\mathrm{ant}}\cap G_{\mathrm{aff}}} &= \\ \big(\mathcal{O}(G_{\mathrm{ant}}) \otimes \mathcal{O}(M_{\mathrm{aff}})\big)^{G_{\mathrm{ant}}\cap G_{\mathrm{aff}}} &= \\ \mathcal{O}(M_{\mathrm{aff}})^{G_{\mathrm{ant}}\cap G_{\mathrm{aff}}}&, \end{aligned}$$

where for the last equality we used that $\mathcal{O}(G_{\mathrm{ant}}) = \Bbbk$. □

Corollary 3.2. *Assume that* $\mathrm{char}(\Bbbk) = p > 0$. *Then $\mathcal{O}(M)$ is a finitely generated algebra.*

Proof. Indeed, by Proposition 2.11, it follows that $(G_{\rm aff}\cap G_{\rm ant})^0$, the connected component of the identity, is a torus, and hence $\mathcal{O}(M)=\mathcal{O}(M_{\rm aff})^{G_{\rm aff}\cap G_{\rm ant}}$ is a finitely generated algebra. □

Theorem 3.3. *Let M be a normal algebraic monoid and let $e\in E(M)$ be a central idempotent. Let $M_e=\{m\in M : me=e\}$ and let $G_e=\{g\in G(M) : ge=e\}$. Then (1) $M_e=\overline{G_e}$, and M_e is an irreducible algebraic monoid, with unit group $G_e\subset G_{\rm aff}$. In particular, G_e and M_e are affine.*

(2) The subset $eM\subset M$ is an algebraic monoid, closed in M. The morphism $\ell_e : M\to eM$, $m\mapsto em$, is a morphism of algebraic monoids, with $M_e=\ell_e^{-1}(1)$.

(3) The unit group of eM equals $G(eM)=eG$. Moreover, $(eG)_{\rm aff}=e(G_{\rm aff})$ and $(eG)_{\rm ant}=(eG_{\rm ant})$. In particular, the Chevalley decompositions of eM and $eG=G(eM)$ fit into the following commutative diagram of exact sequences

$$\begin{tikzcd}
1 \arrow[r] & eM_{\rm aff} \arrow[r] & eM \arrow[r, "p"] & A(G) \arrow[r] & 0 \\
1 \arrow[r] & eG_{\rm aff} \arrow[r] \arrow[u, hook] & eG \arrow[r, "p"'] \arrow[u, hook] & A(G) \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

where $eM_{\rm aff}=\overline{eG_{\rm aff}}=e\overline{G_{\rm aff}}$, and $eM=\overline{eG}$.

(4) $eM=G_{\rm ant}eM_{\rm aff}\cong G_{\rm ant}*_{G_{\rm aff}\cap G_{\rm ant}}eM_{\rm aff}\cong eG_{\rm ant}*_{eG_{\rm aff}\cap eG_{\rm ant}}eM_{\rm aff}$.

Proof. (1) Since M is normal at e, it follows by [4, Corollary 2.2.5] that M_e is an irreducible algebraic monoid, with unit group G_e.

Since $M\cong G*_{G_{\rm aff}}M_{\rm aff}$, it follows that if $ge=e$, then $[g,e]=[1,e]$, and hence $g\in G_{\rm aff}$, i.e. $G_e\subset G_{\rm aff}$.

(2) Since $eM=\{x\in M : xe=e\}$, it is clear that eM is a closed subset. Hence, eM is an algebraic monoid and $\ell_e : M\to eM$, $\ell_e(m)=em$, is a morphism of algebraic monoids.

(3) Since $\ell_e : M\to eM$ is a surjective morphism of algebraic monoids, it follows that $G(eM)=eG$. In particular, $G\to eG$ is a surjective morphism of algebraic groups, and hence, by [5, Lemma 1.5], $eG_{\rm ant}\subset(eG)_{\rm ant}$. It is clear that $eG_{\rm aff}\subset(eG)_{\rm aff}$ and $(eG_{\rm aff})(eG_{\rm ant})=eG$. Hence, $eG/eG_{\rm ant}\cong eG_{\rm aff}/(eG_{\rm aff}\cap eG_{\rm ant})$ is an affine algebraic group, since $eG_{\rm aff}\cap eG_{\rm ant}$ is a central subgroup of $eG_{\rm aff}$. It follows that $(eG)_{\rm ant}\subset eG_{\rm ant}$. On the other hand, it is clear that $eG_{\rm aff}$ is a normal subgroup of eG, and the morphism $G\to eG/eG_{\rm aff}$, $g\mapsto eg(eG_{\rm aff})$, induces a surjective morphism $G/G_{\rm aff}\to eG/eG_{\rm aff}$. It follows that $eG/eG_{\rm aff}$ is an abelian variety and hence $eG_{\rm aff}=(eG)_{\rm aff}$.

Finally, since $eG \cong G/G_e$, and $G_e \subset G_{\mathrm{aff}}$, it follows that $\mathcal{A}(eG) = eG/(eG_{\mathrm{aff}}) \cong G/G_{\mathrm{aff}} = \mathcal{A}(G)$, with the Albanese morphism fitting into the following commutative diagram

$$\begin{tikzcd} G \arrow[rr, "\ell_e"] \arrow[d, "\alpha_{eG}"'] & & eG \arrow[d, "\alpha_{eG}"] \\ \mathcal{A}(G) = G/G_{\mathrm{aff}} & & \mathcal{A}(eG) = eG/(eG)_{\mathrm{aff}} \arrow[ll, "\varphi"] \end{tikzcd}$$

where $\varphi : eG/(eG)_{\mathrm{aff}} \cong (G/G_e)/(G/G_e)_{\mathrm{aff}} \to G/G_{\mathrm{aff}}$ is the canonical isomorphism obtained by observing that $(G/G_e)_{\mathrm{aff}} = G_{\mathrm{aff}}/G_e$.

(4) follows from the description above and Proposition 2.12. □

Remark 3.4. Let M be a normal affine algebraic monoid and let $e \in E(M)$ be a central idempotent. Then $\ell_e : M \to M$, $\ell_e(m) = em$, is such that $\ell_e \circ \ell_e = \ell_e$. In other words, $\varphi^* : \mathcal{O}(eM) \to \mathcal{O}(M)$ is a section for the canonical surjection $\mathcal{O}(M) \to \mathcal{O}(eM)$.

Corollary 3.5. *Let M be a normal algebraic monoid and let $e \in E(M)$ be a central idempotent. Then $\mathcal{O}(M) = \mathcal{O}(eM) \oplus \mathcal{I}(eM_{\mathrm{aff}})^{G_{\mathrm{ant}} \cap G_{\mathrm{aff}}}$. In particular, if $e \in E\big(\overline{G_{\mathrm{ant}} \cap G_{\mathrm{aff}}}\big)$, then $\mathcal{O}(M) = \mathcal{O}(eM)$.*

Proof. Indeed, by Theorem 3.1, it follows that

$$\begin{aligned} \mathcal{O}(M) = \mathcal{O}(M_{\mathrm{aff}})^{G_{\mathrm{ant}} \cap G_{\mathrm{aff}}} = & \\ & \big(\mathcal{O}(eM_{\mathrm{aff}}) \oplus \mathcal{I}(eM_{\mathrm{aff}})\big)^{G_{\mathrm{ant}} \cap G_{\mathrm{aff}}} = \\ & \mathcal{O}(eM_{\mathrm{aff}})^{G_{\mathrm{ant}} \cap G_{\mathrm{aff}}} \oplus \mathcal{I}(eM_{\mathrm{aff}})^{G_{\mathrm{ant}} \cap G_{\mathrm{aff}}} = \\ & \mathcal{O}(eM) \oplus \mathcal{I}(eM_{\mathrm{aff}})^{G_{\mathrm{ant}} \cap G_{\mathrm{aff}}}. \end{aligned}$$

Assume now that $e \in E\big(\overline{G_{\mathrm{ant}} \cap G_{\mathrm{aff}}}\big)$ and let $f \in \mathcal{O}(eM_{\mathrm{aff}})^{G_{\mathrm{ant}} \cap G_{\mathrm{aff}}}$. If $x \in M$, then $f(x) = f(g \cdot x)$ for all $g \in G_{\mathrm{ant}} \cap G_{\mathrm{aff}}$. It follows that $f(x) = f(ex)$, since $e \in \overline{G_{\mathrm{ant}} \cap G_{\mathrm{aff}}}$. In particular, if $f \in \mathcal{I}(eM_{\mathrm{aff}})^{G_{\mathrm{ant}} \cap G_{\mathrm{aff}}}$, then $f(x) = f(ex) = 0$ for all $x \in M$. □

Remark 3.6. The reader should notice that

$$E\big(\overline{G_{\mathrm{ant}} \cap G_{\mathrm{aff}}}\big) = E\big(\overline{G_{\mathrm{ant}}}\big).$$

Indeed, it follows from [5, Prop. 3.1] that $(G_{\mathrm{ant}})_{\mathrm{aff}} \subset G_{\mathrm{aff}} \cap G_{\mathrm{ant}}$. Since $E(N) \subset N_{\mathrm{aff}}$ for any algebraic monoid ([6, Cor. 2.4]), it follows that $E\big(\overline{G_{\mathrm{ant}}}\big) \subset E\big(\overline{G_{\mathrm{ant}} \cap G_{\mathrm{aff}}}\big)$, the other inclusion being obvious.

Definition 3.7. An algebraic monoid M is *anti-affine* if $\mathcal{O}(M) = \Bbbk$.

Let M be an algebraic monoid such that $G(M)$ is an anti-affine algebraic group. Then M is anti-affine. The converse is not true, as the following example shows.

Example 3.8. Let $T = \Bbbk^* \times \Bbbk^*$ be an algebraic torus of dimension 2, and consider the affine toric variety $T \subset \mathbb{A}^2$. Then $\mathbb{A}^2$ is an affine algebraic monoid with unit group T. Let A be a non-trivial connected abelian variety and consider an extension

$$0 \longrightarrow \Bbbk^* = T_1 \longrightarrow H \longrightarrow A \longrightarrow 0$$

of algebraic groups such that H is anti-affine. Let $G = (T \times H)/T_1$, where $T_1 \hookrightarrow T \times H$, $t \mapsto (t,t)$. Then $G_{\mathrm{aff}} = T$, $G_{\mathrm{ant}} = H$, and $G_{\mathrm{aff}} \cap G_{\mathrm{ant}} = T_1 = \big\{(t,t) : t \in \Bbbk^*\big\} \subset T$.

The quotient $M = (\mathbb{A}^2 \times H)/T_1$ is an algebraic monoid with unit group G and with $M_{\mathrm{aff}} \cong \mathbb{A}^2$. Thus

$$\mathcal{O}(M) = \mathcal{O}(M_{\mathrm{aff}})^{G_{\mathrm{aff}} \cap G_{\mathrm{ant}}} = \Bbbk[x,y]^{T_1} = \Bbbk.$$

Hence, M is an anti-affine algebraic monoid while $G(M)$ is not an anti-affine algebraic group.

Definition 3.9. Let G be an algebraic group and X be a G-variety. We say that the action is *generically stable* (equivalently that X is a *generically stable G-variety*) if there exists an open subset consisting of closed orbits. We say that an algebraic monoid M is *stable* if it is generically stable as a $(G_{\mathrm{aff}} \cap G_{\mathrm{ant}})$-variety.

Remark 3.10. (1) Recall that any regular action is such that there exists an open subset of orbits of maximal dimension. Hence, an action $G \times X \to X$ is generically stable if and only if the set of orbits of maximal dimension contains an open subset consisting of closed orbits.
(2) If M is an algebraic monoid, then M is stable if and only if $G_{\mathrm{aff}} \cap G_{\mathrm{ant}}$ is closed in M. Indeed, the coset $g(G_{\mathrm{aff}} \cap G_{\mathrm{ant}})$, $g \in G$, is closed in M if and only if $G_{\mathrm{aff}} \cap G_{\mathrm{ant}}$ is closed in M.

Definition 3.11. Let G be an affine algebraic group acting on an affine variety X. We say that the action is *observable* if for every non-zero G-stable ideal $I \subset \mathcal{O}(X)$, $I^G \neq (0)$. Here we consider the induced action $G \times \mathcal{O}(X) \to \mathcal{O}(X)$, $(a \cdot f)(x) = f(a^{-1}x)$, for all $a \in G$, $x \in X$, $f \in \mathcal{O}(X)$.

The concept of observable action is a generalization of the notion of observable subgroup. Observable subgroups were introduced by Bialynicki-Birula, Hochschild and Mostow in [2] and have been researched extensively since then, notably by F. Grosshans (see [10] for a survey on this topic). Given an affine algebraic group G, a closed subgroup $H \subset G$ is said to be *observable* if G/H is a quasi-affine algebraic variety. The equivalent definition of H being observable if every nonzero H-stable ideal $I \subset \mathcal{O}(G)$ has the property that $I^G \neq (0)$, was first recorded in [9], and then further generalized in [13]. We present here some of the basic results that we need in what follows. We include some of the proofs here for convenience.

Theorem 3.12. *Let G be an affine group acting on an affine variety X. Then the action is observable if and only if (1) $\big[\mathcal{O}(X)\big]^G = \big[\mathcal{O}(X)^G\big]$ and (2) $\Omega(X)$ contains a nonempty open subset.*

Proof. We will only prove that if the action is observable then conditions (1) and (2) hold. We refer the reader to [13, Thm. 3.10] for a complete proof.

Clearly $\big[\mathcal{O}(X)^G\big] \subset \big[\mathcal{O}(X)\big]^G$. Let $g \in \big[\mathcal{O}(X)\big]^G$, and consider the ideal $I = \big\{ f \in \mathcal{O}(X) : fg \in \mathcal{O}(X)\big\}$. Clearly I is G-invariant, and hence there exists $f \in \mathcal{O}(X)^G$ such that $fg \in \mathcal{O}(X)^G$.

Let now $X_{\max}$ be the (open) subset of orbits of maximal dimension and let $Y = X \setminus X_{\max}$. Let $0 \neq f \in \mathcal{I}(Y)^G$. Then the affine open subset X_f is is G-stable subset contained in $X_{\max}$. Let $\mathcal{O} \subset X_f$ be an orbit. Since every orbit is open in its closure, it follows that if $\overline{\mathcal{O}} \neq \mathcal{O}$ then $\overline{\mathcal{O}} \cap Y \neq \emptyset$. Since f is constant on the orbits, it follows that $f|_{\mathcal{O}} = 0$ and hence $\mathcal{O} \subset Y$, that is a contradiction. □

In [13] the reader can find examples showing that both conditions (1) and (2) of Theorem 3.12 are necessary. However, for some families of algebraic groups, generic stability of the action implies observability. This is the case when G is a reductive group (see [13, Thm. 4.7]), or when $G = U \times L$, where L is reductive and U is unipotent.

Proposition 3.13. *Let G be an affine algebraic group with Levi decomposition $G = L \times U$, as above. Assume that X is a generically stable affine G-variety. Then the action is observable.*

Proof. Let $I \subsetneq \mathcal{O}(X)$ be a non-zero G-stable ideal. Then $\mathcal{V}(I) \neq \emptyset$ is a proper closed subset. Since there exists a open subset of closed orbits, then there exists a closed orbit Z such that $Z \cap \mathcal{V}(I) = \emptyset$. Since L is reductive and that Z is L-stable, it follows that there exists $f \in \mathcal{O}(X)^L$ such that $f \in I$, $f|_Z = 1$. Hence, $I^L \neq (0)$. Since U normalizes L, it follows that $I \cap \mathcal{O}(X)^L \neq \{0\}$ is an U-submodule, and hence $I^G = (I^L)^U \neq (0)$. □

Proposition 3.14. *Let M be a stable algebraic monoid, with $G(M)$ an anti-affine algebraic group. Then $M = G(M)$.*

Proof. If $G = G(M)$ is anti-affine, then $G_{\mathrm{aff}} = G_{\mathrm{aff}} \cap G_{\mathrm{ant}}$, and since M is stable, it follows that $M_{\mathrm{aff}} = \overline{G_{\mathrm{aff}}} = \overline{G_{\mathrm{aff}} \cap G_{\mathrm{ant}}} = G_{\mathrm{aff}} \cap G_{\mathrm{ant}} = G_{\mathrm{aff}}$. Hence $M = GM_{\mathrm{aff}} = GG_{\mathrm{aff}} = G$. □

Theorem 3.15. *Let M be a stable normal algebraic monoid. Then $\big[\mathcal{O}(M)\big] = \big[\mathcal{O}(G)\big]$.*

Proof. Theorem 3.1 guarantees that $\big[\mathcal{O}(M)\big] = \big[\mathcal{O}(M_{\mathrm{aff}})^{G_{\mathrm{aff}} \cap G_{\mathrm{ant}}}\big]$. Since M is stable then, by Proposition 3.13 and Theorem 3.12, it follows

that $\big[\mathcal{O}(M)\big] = \big[\mathcal{O}(M_{\mathrm{aff}})\big]^{G_{\mathrm{aff}}\cap G_{\mathrm{ant}}}$. But M_{aff} and G_{aff} being affine varieties, it follows that $\big[\mathcal{O}(M)\big] = \big[\mathcal{O}(G_{\mathrm{aff}})\big]^{G_{\mathrm{aff}}\cap G_{\mathrm{ant}}}$. Now, applying the same reasoning to the algebraic group G, we obtain that $\big[\mathcal{O}(M)\big] = \big[\mathcal{O}(G_{\mathrm{aff}})^{G_{\mathrm{aff}}\cap G_{\mathrm{ant}}}\big] = \big[\mathcal{O}(G)\big]$. $\square$

Corollary 3.16. *Let M be an stable monoid. If M is anti-affine, then M is an (anti-affine) algebraic group.*

Proof. By Theorem 3.15, it follows that $G = G(M)$ is an anti-affine algebraic group. Hence, by Proposition 3.14, it follows that $M = G$. $\square$

Proposition 3.17. *Let M be a normal algebraic monoid and let $e \in E(M)$ be a central idempotent. Then eM is stable if and only if $ef = e$ for all $f \in E\big(\overline{G_{\mathrm{ant}}}\big)$. In particular, eM is stable if and only if $ef_0 = e$, where $f_0 \in E\big(\overline{G_{\mathrm{ant}}}\big)$ is the minimum idempotent.*

Proof. We begin by recalling that $E\big(\overline{G_{\mathrm{ant}}}\big) = E\big(\overline{G_{\mathrm{aff}} \cap G_{\mathrm{ant}}}\big)$ (see Remark 3.6). Let $e \in E(M)$ be a central idempotent. It follows from Theorem 3.3 that $G(eM) = eG$, and that $(eG)_{\mathrm{aff}} \cap (eG)_{\mathrm{ant}} = e(G_{\mathrm{aff}} \cap G_{\mathrm{ant}})$. Since $\ell_e|_{\overline{G_{\mathrm{aff}}\cap eG_{\mathrm{ant}}}} : \overline{G_{\mathrm{aff}} \cap G_{\mathrm{ant}}} \to \overline{e(G_{\mathrm{aff}} \cap G_{\mathrm{ant}})}$ is a surjective morphism of algebraic monoids, it follows from [12, Corollary 3.9] that

$$E\big(\overline{(eG)_{\mathrm{aff}} \cap (eG)_{\mathrm{ant}}}\big) = eE\big(\overline{G_{\mathrm{aff}} \cap G_{\mathrm{ant}}}\big),$$

Since $e \in eE\big(\overline{G_{\mathrm{aff}} \cap G_{\mathrm{ant}}}\big)$, it follows that eM is stable if and only if $ef = e$ for all $f \in E\big(\overline{G_{\mathrm{aff}} \cap G_{\mathrm{ant}}}\big)$.

Let now $G_{\mathrm{aff}}\cap G_{\mathrm{ant}} = TU$ be a Levi decomposition. Since $G_{\mathrm{aff}}\cap G_{\mathrm{ant}}$ is commutative, it follows that $E\big(\overline{G_{\mathrm{aff}} \cap G_{\mathrm{ant}}}\big) = E\big(\overline{T}\big)$. Indeed, if follows from [12, Prop. 3.13] that $E\big(\overline{G_{\mathrm{aff}} \cap G_{\mathrm{ant}}}\big) = \bigcup_{g\in G_{\mathrm{aff}}\cap G_{\mathrm{ant}}} gE(\overline{T})g^{-1} = E(\overline{T})$. Let $f_0 \in \overline{T}$ the unique minimal idempotent; i.e. f_0 is unique idempotent in the Kernel of the affine toric variety $\overline{T}$ — the existence of f_0 follows from the fact that there exists a finite number of idempotents elements on $\overline{T}$. It is clear that $ef_0 = e$ if and only if $ef = e$ for all $f \in E\big(\overline{G_{\mathrm{aff}} \cap G_{\mathrm{ant}}}\big)$. $\square$

Remark 3.18. Observe that, if we keep the notations of the proof of Proposition 3.17, then f_0 is the unique idempotent in $E\big(\overline{G_{\mathrm{ant}}}\big)$ such that f_0M is an stable monoid. Moreover, f_0 is the maximum central idempotent for which f_0M is a stable monoid.

We now characterize normal, anti-affine, algebraic monoids.

Theorem 3.19. *Let M be a normal algebraic monoid. Then M is anti-affine if and only if eM is an anti-affine algebraic group, where e is the unique minimal idempotent of $\overline{G_{\mathrm{ant}}}$ (equivalently, e is the unique minimal idempotent of $\overline{G_{\mathrm{aff}} \cap G_{\mathrm{ant}}}$). In particular, e is the* minimum *idempotent of M.*

Proof. Firs of of, we recall that $E\big(\overline{G_{\mathrm{aff}}\cap G_{\mathrm{ant}}}\big) = E\big(\overline{G_{\mathrm{ant}}}\big)$ (see Remark 3.6) and that $\mathcal{O}(M)=\mathcal{O}(eM)$ (see Corollary 3.5). Since by Proposition 3.17 eM is an stable algebraic monoid, if follows from Corollary 3.16 that $\mathcal{O}(eM)=\Bbbk$ if and only if eM is an anti-affine algebraic group.

Finally, since eM is a group, it follows that $E(eM)=e$, and hence $ef=e$ for $f\in E(M)$. □

The following examples indicate why, in Theorem 3.19, one must focus on the idempotents of $E\big(\overline{G_{\mathrm{aff}}\cap G_{\mathrm{ant}}}\big)$, rather than just any central idempotent.

Example 3.20. (1) Let N be an affine monoid of dimension $\dim N\geq 1$ with zero element 0_N. Then 0_N is a central idempotent of N. Let H be any anti-affine algebraic group. Then $M=N\times H$ is such that $\mathcal{O}(M)=\mathcal{O}(N)$, while $0_NM=H$ is an anti-affine algebraic group. Observe that 0_N is the minimum idempotent of M.

Here, $M_{\mathrm{aff}}=N\times H_{\mathrm{aff}}$, $G(M)_{\mathrm{aff}}=G(N)\times H_{\mathrm{aff}}$ and $G(M)_{\mathrm{ant}}=\{1\}\times H_{\mathrm{ant}}$. Hence $G(M)_{\mathrm{aff}}\cap G(M)_{\mathrm{ant}}=\{1\}\times H_{\mathrm{aff}}$, and thus $0\notin E\big(\overline{G(M)_{\mathrm{aff}}\cap G(M)_{\mathrm{ant}}}\big)$.

(2) Let N be an irreducible affine monoid, with zero element 0_N, such that $0_N\notin\overline{\mathcal{Z}(G)}$ (take for example N as in example 2.7). Let M be an algebraic monoid such that $M_{\mathrm{aff}}=N$. Then, $0_NM=0_NG(M)_{\mathrm{ant}}\cong G(M)_{\mathrm{ant}}/\big(G(M)_{\mathrm{ant}}\big)_{0_N}$ is an anti-affine algebraic group (see for example [5, Lemma 1.3]), whereas $\mathcal{O}(M)=\mathcal{O}(N)^{G(N)\cap G(M)_{\mathrm{ant}}}$ is not necessarily equal to the field $\Bbbk$.

We now show that if M is anti-affine, then $\ell_e : M\to eM$ is Serre's universal morphism from the pointed variety (M,e) into a commutative algebraic group. See [17, Thm. 8] and [5, §2.4] for some basic properties of this morphism.

Theorem 3.21. *Let M be a normal anti-affine algebraic monoid, and let $e\in E(M)$ be its minimum idempotent. Then $\ell_e : M\to eM$ is Serre's universal morphism from the pointed variety (M,e) into a commutative algebraic group. In particular, Serre's morphism fits into the following short exact sequence of algebraic monoids.*

$$1\longrightarrow M_e\longrightarrow M\longrightarrow eM\longrightarrow 1$$

Proof. Let $\sigma : M\to S$ be Serre's universal morphism from the pointed variety (M,e) into a commutative algebraic group. Since M is anti-affine, it follows that S is necessarily an anti-affine algebraic group (see for example [5, §2.4]). Moreover, it follows from Theorem 3.19 that eM is an anti-affine algebraic group. Thus we have a commutative diagram

$$\begin{tikzcd} M \arrow[r, "\ell_e"] \arrow[d, "\sigma"'] & eM \\ S \arrow[ur, "\varphi"'] & \end{tikzcd}$$

Since $\varphi(0_S) = e$, it follows that φ is a morphism of algebraic groups (see for example [5, Lem.1.5]). Consider the associated short exact sequence

$$0 \longrightarrow N \longrightarrow S \xrightarrow{\varphi} eM \longrightarrow 0 \tag{3.1}$$

Since eM is anti-affine with $\sigma(e) = 0_S$, it follows that $\sigma|_{eM} : eM \to S$ is a morphism of algebraic groups. Moreover, we have the following commutative diagram

$$\begin{tikzcd} eM \arrow[rr] \arrow[d, "\alpha_{eM}"'] & & M \arrow[r, two heads, "\sigma"] \arrow[d, "\alpha_M"] & S \arrow[d, "\alpha_S"] \\ \mathcal{A}(eM) = \mathcal{A}(G) \arrow[rr] & & \mathcal{A}(M) = \mathcal{A}(G) \arrow[r, two heads, "\alpha_\sigma"'] & \mathcal{A}(S) \end{tikzcd}$$

Let $\gamma : \mathcal{A}(eM) \to \mathcal{A}(S)$ be the composition of the horizontal arrows. Then γ is a surjective morphism of algebraic groups. In particular, $\dim \mathcal{A}(S) \leq \dim \mathcal{A}(eM)$.

On the other hand, $\varphi \circ \sigma = \ell_e : M \to eM$, and thus $\sigma|_{eM}$ is a splitting for the exact sequence (3.1). It follows that $S \cong eM \times N$, and we have the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & eM \arrow[r] \arrow[d, "\alpha_{eM}"'] & S = eM \times N \arrow[r] \arrow[d, "\alpha_S"] & N \arrow[r] \arrow[d, "\alpha_N"] & 0 \\ & \mathcal{A}(eM) \arrow[r, "\gamma"'] & \mathcal{A}(S) \arrow[r, "\beta"'] & \mathcal{A}(N) \arrow[r] & 0 \end{tikzcd}$$

It follows that $\gamma\big(\mathcal{A}(eM)\big) \subset \beta^{-1}(0)$. Hence,

$$\dim \mathcal{A}(S) \leq \dim \mathcal{A}(eM) \leq \dim \mathcal{A}(S) - \dim \mathcal{A}(N).$$

where the second inequality follows from Chevalley's theorem on the dimension of the fibers of a morphism. Thus, equality holds and $\dim \mathcal{A}(N) = 0$. It follows that N is an irreducible affine algebraic group. Since $\Bbbk = \mathcal{O}(S) \cong \mathcal{O}(eM) \otimes \mathcal{O}(N) \cong \mathcal{O}(N)$, it follows that N is a point, and thus $S \cong eM$. $\square$

We conclude this paper by extending the Rosenlicht decomposition of algebraic groups ([5, Prop. 3.1]), to the setting of algebraic monoids.

Theorem 3.22. *Let M be a normal algebraic monoid, with unit group G, and let $e \in E\big(\overline{G_{\mathrm{ant}}}\big)$ be the minimum idempotent of $\overline{G_{\mathrm{ant}}}$. Assume*

that $\mathcal{O}(M)$ is finitely generated. Then $M_e G_{\mathrm{ant}}$ is an anti-affine algebraic monoid, and the sequence

$$1 \longrightarrow M_e G_{\mathrm{ant}} \longrightarrow M \xrightarrow[\varphi]{} \operatorname{Spec}\big(\mathcal{O}(M)\big)$$

is an exact sequence of algebraic monoids, with φ a dominant morphism.

Proof. Since $\mathcal{O}(M)$ is finitely generated, it follows that $N = \operatorname{Spec}\big(\mathcal{O}(M)\big)$ is an algebraic monoid. Moreover, the canonical morphism $\varphi : M \to N$ is a morphism of algebraic monoids. If we let $S = \varphi^{-1}(1)$, then we have the following commutative diagram. The top row is exact and the bottom row is left exact.

$$\begin{tikzcd}
1 \arrow[r] & G_{\mathrm{ant}} \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, hook] & \operatorname{Spec}\big(\mathcal{O}(G)\big) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & S \arrow[r] & M \arrow[r, "\varphi"'] & N = \operatorname{Spec}\big(\mathcal{O}(M)\big) &
\end{tikzcd}$$

Since $M = G_{\mathrm{ant}} M_{\mathrm{aff}} \cong G_{\mathrm{ant}} *_{G_{\mathrm{aff}} \cap G_{\mathrm{ant}}} M_{\mathrm{aff}}$ (Proposition 2.12) and $\mathcal{O}(M) \cong \mathcal{O}(M_{\mathrm{aff}})^{G_{\mathrm{aff}} \cap G_{\mathrm{ant}}}$ (Theorem 3.1), it follows that the restriction morphism $\varphi|_{M_{\mathrm{aff}}}$ is induced by the inclusion $\mathcal{O}(M_{\mathrm{aff}})^{G_{\mathrm{aff}} \cap G_{\mathrm{ant}}} \subset \mathcal{O}(M_{\mathrm{aff}})$. In particular, $\varphi|_{M_{\mathrm{aff}}}$ is dominant and the following diagram is commutative

$$\begin{tikzcd}
1 \arrow[r] & S \arrow[r] & M \arrow[r, "\varphi"] & N = \operatorname{Spec}\big(\mathcal{O}(M)\big) \arrow[d, equal] \\
 & G_{\mathrm{aff}} \cap G_{\mathrm{ant}} \arrow[r, hook] \arrow[u, hook] & M_{\mathrm{aff}} \arrow[r] \arrow[u, hook] & N = \operatorname{Spec}\big(\mathcal{O}(M_{\mathrm{aff}})^{G_{\mathrm{aff}} \cap G_{\mathrm{ant}}}\big)
\end{tikzcd}$$

Assume that M is a stable algebraic monoid. Then $G_{\mathrm{aff}} \cap G_{\mathrm{ant}}$ is a commutative closed normal subgroup of M_{aff}, and hence it follows from Proposition 3.13 that $G_{\mathrm{aff}} \cap G_{\mathrm{ant}}$ is observable in M_{aff}. It follows from [13, Thm.3.18] that the *affinized quotient* $\varphi|_{M_{\mathrm{aff}}} : M_{\mathrm{aff}} \to \operatorname{Spec}\big(\mathcal{O}(M_{\mathrm{aff}})^{G_{\mathrm{aff}} \cap G_{\mathrm{ant}}}\big) = N$ is such that there exists an open subset $U \subset N$ so that $(\varphi|_{M_{\mathrm{aff}}})^{-1}(u)$ is a closed $G_{\mathrm{aff}} \cap G_{\mathrm{ant}}$-orbit for all $u \in U$. We claim that this implies that $(\varphi|_{M_{\mathrm{aff}}})^{-1}(1_N) = G_{\mathrm{aff}} \cap G_{\mathrm{ant}}$. Indeed, observe that since $G_{\mathrm{aff}} \cap G_{\mathrm{ant}}$ is normal in M, it follows that $\varphi|_{M_{\mathrm{aff}}}$ is $(G \times G)$-equivariant. It suffices now to recall that $M = G_{\mathrm{ant}} M_{\mathrm{aff}}$, and thus

$$\varphi^{-1}(1_N) = G_{\mathrm{ant}} (\varphi|_{M_{\mathrm{aff}}})^{-1}(1_N) = G_{\mathrm{ant}}.$$

Since M is stable, $e = 1$, and thus $M_e G_{\mathrm{ant}} = G_{\mathrm{ant}}$.

If M is not a stable monoid, let $e \in E\big(\overline{G_{\mathrm{ant}}}\big)$ be the minimum idempotent. Then by Corollary 3.5 it follows that $\mathcal{O}(M) = \mathcal{O}(eM)$. One then concludes from (the proof of) Corollary 3.5, that $\varphi|_{eM} : eM \to$

$N \cong \operatorname{Spec}\big(\mathcal{O}(eM)\big)$ is the affinization morphism of eM. Recalling that $G(eM)_{\text{ant}} = eG_{\text{ant}}$, it follows that eM is stable that we have an exact sequence

$$1 \longrightarrow eG_{\text{ant}} \longrightarrow eM \xrightarrow[\varphi]{} \operatorname{Spec}\big(\mathcal{O}(eM)\big)$$

that fits into the following commutative diagram of algebraic monoids, where the vertical sequence in the center is exact.

$$\begin{tikzcd}
 & & 1 \arrow[d] & \\
 & & M_e \arrow[dl, hook] \arrow[d] & \\
1 \arrow[r] & S \arrow[r] \arrow[d, "\ell_e"] & M \arrow[r, "\varphi"] \arrow[d, "\ell_e"] & \operatorname{Spec}\big(\mathcal{O}(M)\big) \arrow[d, equal] \\
1 \arrow[r] & eG_{\text{ant}} \arrow[r] & eM \arrow[r, "\varphi"'] \arrow[d] & \operatorname{Spec}\big(\mathcal{O}(eM)\big) \\
 & & 0 &
\end{tikzcd}$$

It follows that $S = \ell_e^{-1}(eG_{\text{ant}}) = M_e G_{\text{ant}}$.

To complete the proof we observe that $G(S) = G_e G_{\text{ant}}$, with $G_e \subset G_{\text{aff}}$. Hence $G(S)_{\text{ant}} = G_{\text{ant}}$. Since e is the minimum idempotent of the closure $\overline{G_{\text{ant}}} \subset M$, it follows that e is also the minimum idempotent of the closure $\overline{G_{\text{ant}}} \subset M_e G_{\text{ant}}$. It suffices now to observe that $e(M_e G_{\text{ant}}) = eG_{\text{ant}}$, and then apply Theorem 3.19. $\square$

References

[1] A. Bialynicki-Birula, *On induced actions of algebraic groups*, Ann. de l'Inst. Fourier, 43 no. 2 (1993), 365-368.

[2] A. Bialynicki-Birula, G. Hochschild, G.D. Mostow, *Extensions of representations of algebraic linear groups*, Amer. J. Math. 85 (1963), 131-144.

[3] M. Brion, *Log homogeneous varieties*, Actas del XVI Coloquio Latinoamericano de Álgebra, 1–39, Revista Matemática Iberoamericana, Madrid, 2007; arXiv: math/0609669.

[4] M. Brion, *The local structure of algebraic monoids*, preprint, arXiv:0709.1255 [math.AG].

[5] M. Brion, *Anti-affine algebraic groups*, preprint, arXiv:0710.5211 [math.AG].

[6] M. Brion, A. Rittatore *The structure of normal algebraic monoids*, Semigroup Forum 74 (2007), no. 3, 410–422; arXiv: math.AG/0610351.

[7] B. Conrad, *A modern proof of Chevalley's theorem on algebraic groups*, J. Ramanujan Math. Soc. **17** (2002), 1–18.

[8] M. Demazure and P. Gabriel, *Groupes algébriques*, North Holland, Amsterdam, 1970.

[9] W. Ferrer-Santos, A. Rittatore, *Actions and Invariants of Algebraic Groups.* Series: Pure and Applied Math., 268, Dekker-CRC Press, Florida, (2005).
[10] F.D. Grosshans, *Observable groups and Hilbert's fourteenth problem*, Amer. J. Math. **95** (1973), 229–253.
[11] M. S. Putcha, *Linear Algebraic Monoids*, London Math. Soc. Lecture Notes Series **133**, Cambridge University Press, Cambridge, 1988.
[12] L. E. Renner, *Linear Algebraic Monoids*, Encyclopædia of Mathematical Sciences **134**, Invariant Theory and Algebraic Transformation Groups, V, Springer-Verlag, Berlin, 2005.
[13] L. E. Renner, A. Rittatore, *Observable actions of algebraic groups*, preprint, arXiv:0902.0137v2 [math.AG].
[14] A. Rittatore, *Algebraic monoids and group embeddings*, Transformation Groups **3**, No. 4 (1998), 375–396; arXiv: math.AG/9802073.
[15] A. Rittatore, *Algebraic monoids with affine unit group are affine*, Transform. Groups **12** (2007), 601–605; arXiv: math.AG/0602221.
[16] M. Rosenlicht, *Some basic theorems on algebraic groups*, Amer. J. Math. **78** (1956), 401–443.
[17] J.-P. Serre, *Morphismes universels et variété d'Albanese*, Séminaire Chevalley (1958–1959), Exposé No. 10, Documents Mathématiques **1**, Soc. Math. France, Paris, 2001.
[18] L. Solomon, *An introduction to reductive monoids*, in Semigroups, formal languages and groups, York, 1993, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 466, 1995, 295–352.
[19] D. A. Timashev, *Homogeneous spaces and equivariant embeddings*, to appear in the Encyclopædia of Mathematical Sciences, subseries Invariant Theory and Algebraic Transformation Groups; arXiv: math.AG/0602228.

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